GRAHAM POSITIVITY OF QUANTUM DOUBLE SCHUBERT POLYNOMIALS

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Abstract. We prove a positivity property for coefficients in the product expansion of quantum double Schubert polynomials.

1. Introduction

Let $S_\infty = \bigcup_{m\geq 1} S_m$, $\mathbf{x} = (x_1, x_2, \ldots)$, $\mathbf{y} = (y_1, y_2, \ldots)$, $\mathbf{t} = (t_1, t_2, \ldots)$ and $\mathbf{q} = (q_1, q_2, \ldots)$. Since $\mathbb{Z}[\mathbf{x}, \mathbf{t}][\mathbf{q}] = \bigoplus_{w\in S_\infty} \mathbb{Z}[\mathbf{t}][\mathbf{q}]\mathfrak{S}^q_w(\mathbf{x}; \mathbf{t})$, we can consider the following product expansion by extending scalars from $\mathbb{Z}[\mathbf{t}][\mathbf{q}]$ to $\mathbb{Z}[\mathbf{y}, \mathbf{t}][\mathbf{q}]$:

$$\mathfrak{S}^q_u(\mathbf{x}; \mathbf{y})\, \mathfrak{S}^q_v(\mathbf{x}; \mathbf{t}) = \sum_{w\in S_\infty} C^w_{u,v}(\mathbf{y}, \mathbf{t}; \mathbf{q})\, \mathfrak{S}^q_w(\mathbf{x}; \mathbf{t}). \tag{1}$$

Claim. *For any $u, v, w \in S_\infty$,*

$$C^w_{u,v}(\mathbf{y}, \mathbf{t}; \mathbf{q}) \in \mathbb{N}[t_i - y_j, q_k \mid i, j, k \geq 1].$$

For fixed $u, v \in S_\infty$, since the direct sum decomposition should be finite, we can choose n sufficiently large such that u, v and every permutation occurring in the expansion belong to S_n, while every variable occurring in its coefficients is among $y_1, \ldots, y_n$, $t_1, \ldots, t_n$, and $q_1, \ldots, q_{2n-1}$. Write $\mathbf{x} = (x_1, \ldots, x_{2n})$ and $\mathbf{q} = (q_1, \ldots, q_{2n-1})$. Following the parameter arrangement method of Gao–Xiong [2], set $\mathbf{a} = (t_1, \ldots, t_n, y_1, \ldots, y_n)$, and let $\tau \in S_{2n}$ be the permutation such that $\tau\mathbf{a} = (y_1, \ldots, y_n, t_1, \ldots, t_n)$. (1) can be rewritten as:

$$\mathfrak{S}^q_u(\mathbf{x}; \tau\mathbf{a})\mathfrak{S}^q_v(\mathbf{x}; \mathbf{a}) = \sum_{w\in S_{2n}} C^{w,\tau}_{u,v}(\mathbf{a}; \mathbf{q})\mathfrak{S}^q_w(\mathbf{x}; \mathbf{a}). \tag{2}$$

Following Lam–Shimozono [4, Section 2.1], let e_i and $E^{(2n)}_i$ be the elementary symmetric polynomials and their quantum analogues, and write $J^a_{2n} = \langle e_i(\mathbf{x}) - e_i(\mathbf{a})\rangle$ and $J^{qa}_{2n} = \langle E^{(2n)}_i(\mathbf{x}, \mathbf{q}) - e_i(\mathbf{a})\rangle$. Follow Fomin–Gelfand–Postnikov [1, Theorem 1.1, Proposition 3.3] and extend the map by $\mathbb{Z}[\mathbf{a}]$-linearity to a $\mathbb{Z}[\mathbf{q}, \mathbf{a}]$-module automorphism of $\mathbb{Z}[\mathbf{x}, \mathbf{q}, \mathbf{a}]$, as in Lam–Shimozono [4, Section 3.3]. This induces a map $\bar{\theta} : \mathbb{Z}[\mathbf{x}, \mathbf{a}]/J^a_{2n} \to \mathbb{Z}[\mathbf{x}, \mathbf{a}, \mathbf{q}]/J^{qa}_{2n}$.

Classical equivariant Borel presentation gives an isomorphism between $\mathbb{Z}[\mathbf{x}, \mathbf{a}]/J^a_{2n}$ and $H^*_T(\mathrm{Fl}_{2n})$; Kim's presentation [3, Theorem 2] gives an isomorphism between $\mathbb{Z}[\mathbf{x}, \mathbf{a}, \mathbf{q}]/J^{qa}_{2n}$ and $QH^*_T(\mathrm{Fl}_{2n})$. $\mathbb{Z}[\mathbf{x}, \mathbf{a}, \mathbf{q}]/J^{qa}_{2n} \cong QH^*_T(\mathrm{Fl}_{2n})$; Lam–Shimozono [4, Theorem 5] identifies the basis $[\mathfrak{S}^q_w(\mathbf{x}; \mathbf{a})]_{J^{qa}_{2n}} \longleftrightarrow \sigma^T_w$. ϕ is the canonical $\mathbb{Z}[\mathbf{a}]$-linear embedding sending $[X_w]^T$

to σ_w^T. We have the following commutative diagram:

$$\begin{tikzcd}
\mathbb{Z}[\mathbf{x},\mathbf{a}]/J_{2n}^a \arrow[r, "\rho_{2n}^a", "\cong"'] \arrow[d, "\bar\theta"] \arrow[dr, phantom, "\circlearrowleft"] & H_T^*(\mathrm{Fl}_{2n}) \arrow[d, "\phi"] & {[\mathfrak{S}_w(\mathbf{x};\mathbf{a})]_{J_{2n}^a}} \arrow[r, mapsto] \arrow[d, mapsto] & {[X_w]^T} \arrow[d, mapsto] & {[\mathfrak{S}_u(\mathbf{x};\tau\mathbf{a})]_{J_{2n}^a}} \arrow[r, mapsto, red] \arrow[d, mapsto, blue] & {[\tau X_u]^T} \arrow[d, mapsto, green] \\
\mathbb{Z}[\mathbf{x},\mathbf{a},\mathbf{q}]/J_{2n}^{qa} \arrow[r, "\rho_{2n}^{qa}", "\cong"'] & QH_T^*(\mathrm{Fl}_{2n}) & {[\mathfrak{S}_w^q(\mathbf{x};\mathbf{a})]_{J_{2n}^{qa}}} \arrow[r, mapsto] & \sigma_w^T & {[\mathfrak{S}_u^q(\mathbf{x};\tau\mathbf{a})]_{J_{2n}^{qa}}} \arrow[r, mapsto] & \phi([\tau X_u]^T)
\end{tikzcd}$$

Remark. The red arrow is what Gao–Xiong used in geometric proof [2]; the blue arrow is because of the $\mathbb{Z}[\mathbf{a}]$-linearity of $\bar\theta$; the green arrow records that the twisted quantum Schubert class is $\phi([\tau X_u]^T)$, which allows us to apply Mihalcea's EQLR.

Thus we can convert (2) to a geometric form in the flag variety. Decompose $\mathfrak{S}_u^q(\mathbf{x};\tau\mathbf{a})$ along the quantum Schubert basis, embed the whole polynomial equation to the polynomial classes and map it into $QH_T^*(\mathrm{Fl}_{2n})$, then compare the coefficients of σ_w^T with the definition of EQLR in Mihalcea [5, Section 5]. Following Mihalcea's notations [5], and writing Y_w for the opposite Schubert variety (see Section 2), we have

$$C_{u,v}^{w,\tau}(\mathbf{a};\mathbf{q}) = \sum_{\mathbf{d}\geq 0} \mathbf{q}^{\mathbf{d}} (\pi_{\mathcal{M}_{\mathbf{d}}})_*^T \big(\mathrm{ev}_1^*[\tau X_u]^T \cdot \mathrm{ev}_2^*[X_v]^T \cdot \mathrm{ev}_3^*[Y_w]^T\big). \tag{3}$$

2. Proof of Claim

Let $G = \mathrm{GL}_{2n}$, $B \subset G$ be a Borel subgroup, B^- be the opposite Borel, and $T = B \cap B^-$, so that $X = G/B = \mathrm{Fl}_{2n}$. Following Gao–Xiong [2], let N^- be the unipotent radical of B^-, $w_0^{(n)}$ be the longest element of S_n, and write $N^-(w) = N^- \cap wN^-w^{-1}$, $B^-(w) = TN^-(w)$. For $w \in S_{2n}$ we take the B^--Schubert varieties $X_w = \overline{B^-wB/B}$ and the opposite Schubert varieties $Y_w = \overline{BwB/B}$. For a fixed effective degree $\mathbf{d}$, define $I_{u,v}^{\mathbf{d}} = (\mathrm{ev}_1, \mathrm{ev}_2)^{-1}(\tau X_u \times X_v)$ and $\Gamma_{u,v}^{\mathbf{d}} = (\mathrm{ev}_3)_*[I_{u,v}^{\mathbf{d}}]$.

Proposition 2.1. *Following Mihalcea's notations [5], write $\mathcal{M}_{\mathbf{d}} = \overline{\mathcal{M}}_{0,3}(X,\mathbf{d})$ with evaluation maps ev_i and $F = (\mathrm{ev}_1,\mathrm{ev}_2) : \mathcal{M}_{\mathbf{d}} \to X \times X$. For every $u, v \in S_n$, the incidence scheme $I_{u,v}^{\mathbf{d}} = F^{-1}(\tau X_u \times X_v)$ is reduced and locally irreducible, and each of its irreducible components has codimension $\ell(u) + \ell(v)$. Moreover,*

$$\big(\mathrm{ev}_1^*[\tau X_u]^T \cdot \mathrm{ev}_2^*[X_v]^T\big) \cap [\mathcal{M}_{\mathbf{d}}]^T = [I_{u,v}^{\mathbf{d}}]^T \quad \textit{in } H_*^T(\mathcal{M}_{\mathbf{d}}). \tag{4}$$

Proof. Clearly, F is diagonally G-equivariant. Let $a_0 = 1 \times w_0^{(n)} \in S_{2n}$. Since $u, v \in S_n$, one has

$$a_0 X_u = X_u, \qquad a_0 X_v = X_v, \qquad a_0^{-1}\tau a_0 = w_0^{(2n)}.$$

Thus diagonal left translation by a_0^{-1} identifies

$$\tau X_u \times X_v \;\simeq\; w_0^{(2n)} X_u \times X_v.$$

The moduli space $\mathcal{M}_{\mathbf{d}}$ is irreducible by Thomsen's theorem [6, Theorem 1]. By Mihalcea's incidence lemma [5, Lemma 6.3], $I_{u,v}^{\mathbf{d}}$ is reduced, locally irreducible, and of codimension

$$\mathrm{codim}_X(w_0^{(2n)} X_u) + \ell(v) = \ell(u) + \ell(v).$$

In particular, writing $V = \tau X_u \times X_v \subset X \times X$, every irreducible component of $F^{-1}(V) = I^{\mathbf{d}}_{u,v}$ has codimension $\operatorname{codim}_{X\times X} V = \ell(u)+\ell(v)$ and algebraic multiplicity one. Mihalcea's formula (2") [5, Section 3.1], applied to $F : \mathcal{M}_{\mathbf{d}} \to X \times X$ (with $X \times X$ smooth), therefore yields

$$F^*[V]^T \cap [\mathcal{M}_{\mathbf{d}}]^T = [I^{\mathbf{d}}_{u,v}]^T \quad \text{in } H^T_*(\mathcal{M}_{\mathbf{d}}).$$

Since $F^*[V]^T = \mathrm{ev}_1^*[\tau X_u]^T \cdot \mathrm{ev}_2^*[X_v]^T$, this is precisely (4). □

Lemma 2.2. *The cycle $\Gamma^{\mathbf{d}}_{u,v}$ is an effective $B^-(\tau)$-invariant cycle on X.*

Proof. The variety X_v is B^--invariant. If $h = \tau h'\tau^{-1} \in N^-(\tau) = N^- \cap \tau N^- \tau^{-1}$, then $h(\tau X_u) = \tau h' X_u \subseteq \tau X_u$. Hence $\tau X_u \times X_v$ is $B^-(\tau)$-invariant, and so is its preimage $I^{\mathbf{d}}_{u,v} = F^{-1}(\tau X_u \times X_v)$. By Proposition 2.1, $I^{\mathbf{d}}_{u,v}$ is a reduced closed subscheme of $\mathcal{M}_{\mathbf{d}}$, so its fundamental cycle is effective. Since ev_3 is proper, the push-forward $\Gamma^{\mathbf{d}}_{u,v} = (\mathrm{ev}_3)_*[I^{\mathbf{d}}_{u,v}]$ is an effective cycle on X. Moreover ev_3 is G-equivariant, so $\Gamma^{\mathbf{d}}_{u,v}$ is $B^-(\tau)$-invariant. □

Theorem 2.3. *There is a unique expansion*

$$[\Gamma^{\mathbf{d}}_{u,v}]^T = \sum_{z\in S_{2n}} p^{z,\mathbf{d}}_{u,v}(\mathbf{a})\,[X_z]^T \quad \textit{in } H^*_T(X), \tag{5}$$

and

$$p^{z,\mathbf{d}}_{u,v}(\mathbf{a}) \in \mathbb{N}[-\alpha]_{\alpha\in I(\tau)} = \mathbb{N}[t_i - y_j : 1 \le i,j \le n]. \tag{6}$$

Here $I(\tau) = \{y_j - t_i : 1 \le i,j \le n\}$ is the left inversion set of τ, in the notation of Gao–Xiong [2, Sections 2.1 and 2.3].

Proof. The classes $\{[X_z]^T : z \in S_{2n}\}$ form a $\mathbb{Z}[\mathbf{a}]$-basis of $H^*_T(X)$, so the expansion (5) exists and is unique. Since $\Gamma^{\mathbf{d}}_{u,v}$ is an effective $B^-(\tau)$-invariant cycle on X by Lemma 2.2, apply Gao–Xiong [2, Theorem 2.3 and Corollary 2.4] with $w = \tau$ and $Y = \Gamma^{\mathbf{d}}_{u,v}$ then we obtain the desired expansion. □

Write $b = \mathrm{ev}_1^*[\tau X_u]^T \cdot \mathrm{ev}_2^*[X_v]^T$. By the definition of the equivariant Gysin map [5, Appendix] and (4),

$$(\mathrm{ev}_3)^T_*(b) = \mathrm{PD}\big((\mathrm{ev}_3)_*(b \cap [\mathcal{M}_{\mathbf{d}}]^T)\big) = \mathrm{PD}\big((\mathrm{ev}_3)_*[I^{\mathbf{d}}_{u,v}]^T\big) = [\Gamma^{\mathbf{d}}_{u,v}]^T \qquad \text{in } H^*_T(X), \tag{7}$$

where PD denotes the equivariant Poincaré duality isomorphism (available since X is smooth), and the last equality is the definition $\Gamma^{\mathbf{d}}_{u,v} = (\mathrm{ev}_3)_*[I^{\mathbf{d}}_{u,v}]$. The projection formula [5, Section 3.1, formula (1)] gives $(\mathrm{ev}_3)^T_*(\mathrm{ev}_3^*[Y_w]^T \cdot b) = [Y_w]^T \cdot (\mathrm{ev}_3)^T_*(b)$. Applying $(\pi_X)^T_*$ and using $\pi_{\mathcal{M}_{\mathbf{d}}} = \pi_X \circ \mathrm{ev}_3$ together with (7), we obtain

$$(\pi_{\mathcal{M}_{\mathbf{d}}})^T_* \left(b \cdot \mathrm{ev}_3^*[Y_w]^T\right) = (\pi_X)^T_* \left([\Gamma^{\mathbf{d}}_{u,v}]^T\,[Y_w]^T\right). \tag{8}$$

Combining (3), (5) and (8), we obtain

$$C^{w,\tau}_{u,v}(\mathbf{a};\mathbf{q}) = \sum_{\mathbf{d}\ge 0} \mathbf{q}^{\mathbf{d}} (\pi_X)^T_* \left([\Gamma^{\mathbf{d}}_{u,v}]^T\,[Y_w]^T\right) = \sum_{\substack{\mathbf{d}\ge 0\\ z\in S_{2n}}} p^{z,\mathbf{d}}_{u,v}(\mathbf{a})\mathbf{q}^{\mathbf{d}} (\pi_X)^T_* \left([X_z]^T\,[Y_w]^T\right) = \sum_{\substack{\mathbf{d}\ge 0\\ z\in S_{2n}}} p^{z,\mathbf{d}}_{u,v}(\mathbf{a})\mathbf{q}^{\mathbf{d}}\,\delta_{z,w}.$$

By Theorem 2.3, each $p^{z,\mathbf{d}}_{u,v}(\mathbf{a})$ lies in $\mathbb{N}[t_i - y_j : 1 \le i,j \le n]$, hence $C^{w,\tau}_{u,v}(\mathbf{a};\mathbf{q}) \in \mathbb{N}[t_i - y_j, q_k \mid i,j,k \ge 1]$. This proves the claim.

References

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